Initial Segment Maximal Σ_n-definable Sets in Fragments of Arithmetic

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Abstract

In this paper we will give, for certain models M of some Fragments of Arithmetic, the least initial segment, nonstandard, maximal Σ_n-definable that contains $K_n(M; X)$ with $X \subseteq M$, n.s and not cofinal in M.

1 Preliminaries

$\mathbf{P}^-$ is the theory whose models are the nonnegative parts of the conmutative discretely ordered rings. As usual $\mathbf{I}_\varphi$, $\mathbf{B}_\varphi$ and $\mathbf{L}_\varphi$ are, respectively, the induction, collection and least element axioms for a formula φ of the first order language of Arithmetic, $\mathcal{L}$.

Let $\Gamma \subseteq Form(\mathcal{L})$. Then

$$\mathbf{E}\Gamma = \mathbf{P}^- + \{\mathbf{E}_\varphi : \varphi \in \Gamma\} \quad for \quad \mathbf{E} = \mathbf{I} \text{ or } \mathbf{L}$$
$$\mathbf{B}\Gamma = \mathbf{I}\Delta_0 + \{\mathbf{B}_\varphi : \varphi \in \Gamma\}$$

Peano's Arithmetic, **PA**, is the theory $\mathbf{P}^- + \{\mathbf{I}_\varphi : \varphi \in Form(\mathcal{L})\}$.

Definition 1.1. *Let M_1, M_2 be $\mathcal{L}$-structures such that $M_1 \subset M_2$.*

1. *We say that $X \subseteq M_2$ is an initial segment in M_2 iff it is closed under the succesor function and for all $a \in X$, $b \in M_2$ if $b \leq a$, then $b \in X$.*
2. *We say that $X \subseteq M_2$ is a cofinal set in M_2 iff for all $b \in M_2$ there exists $a \in X$ such that $b \leq a$.*
3. *We say that M_1 is an initial substructure of M_2 ($M_1 \subset^e M_2$) iff M_1 is an initial segment in M_2.*

Definition 1.2. *Let M_1, M_2 be $\mathcal{L}$-structures such that $M_1 \subset M_2$. We say that M_1 is an n-elemental substructure of M_2 ($M_1 \prec_n M_2$) iff for any formula $\varphi(\vec{x}) \in \Sigma_n$ and $\vec{a} \in M_1$*

$$M_1 \models \varphi(\vec{a}) \iff M_2 \models \varphi(\vec{a})$$

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Note: If $M_1 \subset^e M_2$ and $M_1 \prec_n M_2$ we denote $M_1 \prec_n^e M_2$.

Proposition 1.3. *[2] If $M_1 \subset^e M_2$, then $M_1 \prec_0^e M_2$.*

Theorem 1.4 (Tarski-Vaugth Test). *[2] Let $M_1 \prec_0 M_2 \models \mathbf{P}^-$. The following assertions are equivalent:*

1. *$M_1 \prec_{n+1} M_2$.*
2. *For all $\varphi(x,y) \in \Pi_n$ and $a \in M_1$, if $M_2 \models \exists x\, \varphi(x,a)$, then there exists $b \in M_1$ such that $M_2 \models \varphi(b,a)$.*

Theorem 1.5 (Clôte). *[3] If $M_1 \models \mathbf{I}\Sigma_n$ and $M_2 \prec_{n+1}^e M_1$ proper, then $M_2 \models \mathbf{B}\Sigma_{n+2}$.*

Definition 1.6. *Let $M_1 \subset M_2$. The initial segment defined by M_1 in M_2 is the $\mathcal{L}$-structure with universe $S(M_1, M_2) = \{a \in M_2 :$ there exists $b \in M_1$ such that $a \leq b\}$.*

2 Maximal Σ_n-definable Sets

Definition 2.1. *Let M be an $\mathcal{L}$-structure and $X \subseteq M$.*

1. *Let $\varphi(x,\vec{y}) \in \Sigma_n$, $a \in M$ and $\vec{b} \in X$ such that $M \models \varphi(a,\vec{b}) \wedge \forall x\,(\varphi(x,\vec{b}) \rightarrow x = a)$. Then we say that a is Σ_n-definable in M with parameters $\vec{b} \in X$ by the formula φ, and we denote $M \models \varphi(x,\vec{b}) \rightsquigarrow a$.*
2. *$\mathcal{K}_n(M;X) = \{a \in M : a$ is Σ_n-definable in M with parameters in $X\}$.*
3. *$\mathcal{I}_n(M;X) = S(\mathcal{K}_n(M;X), M)$.*

Proposition 2.2. *[2] Let $M \models \mathbf{I}\Sigma_{n+1}$ nonstandard and $X \subseteq M$. If X is not cofinal in M, then $\mathcal{K}_{n+1}(M;X)$ is not cofinal in M.*

Proposition 2.3. *[1] Let $M \models \mathbf{I}\Sigma_n$ and $X \subseteq M$. Then*

1. *$\mathcal{K}_{n+1}(M;X) \prec_{n+1} M$ and $\mathcal{K}_{n+1}(M;X) \models \mathbf{I}\Sigma_n$.*
2. *$\mathcal{I}_{n+1}(M;X) \prec_n^e M$.*

Definition 2.4. *Let M be an $\mathcal{L}$-structure and $X \subseteq M$. We say that X is a maximal Σ_n-definable set in M iff $X \neq M$ and $\mathcal{K}_n(M;X) = X$.*

A first question arises in a natural way.

Question 1. Given a model, M, of a Fragment of Arithmetic, are there initial segments maximal Σ_n-definable in M?

In [5] we give an affirmative answer to these question for models of **PA**. As an example, by means of Tarski-Vaught Test, we give a necessary and sufficient condition for ω to be a maximal Σ_{n+1}-definable set in $M \models \mathbf{I}\Sigma_n$ nonstandard.

Proposition 2.5. *Let $M \models \mathbf{I}\Sigma_n$ be nonstandard. Then*

$$\mathcal{K}_{n+1}(M;\omega) = \omega \iff \omega \prec_{n+1} M$$

Nevertheless, the classic models of Paris-Kirby $\mathcal{K}_n(M;X)$ and $\mathcal{I}_n(M;X)$, do not answer the question, since, in general, the first one is a maximal Σ_n-definable set but not an initial segment and the second one is an initial segment but not a maximal Σ_n-definable set.

As, if $M \models \mathbf{I}\Sigma_{n+1}$ and $X \subseteq M$, finite nonstandard, then $\mathcal{I}_{n+1}(M;X) \models \mathbf{B}\Sigma_{n+1} + \neg \mathbf{I}\Sigma_{n+1}$ [2] two more questions appear.

Question 2. Given a model, M, of a Fragment of Arithmetic, are there models of $\mathbf{B}\Sigma_{n+1}$ and not of $\mathbf{I}\Sigma_{n+1}$ that are initial segments maximal Σ_n-definable in M?

Question 3. Given $X \subseteq M$, which is the least initial segment maximal Σ_n-definable in M containning $\mathcal{K}_n(M;X)$?

In what follows, we will give an answer to these questions.

3 The Structures $T_\Gamma(M;X)$

Definition 3.1. *Let M be an $\mathcal{L}$-structure, $\Gamma \subseteq Form(\mathcal{L})$ and $X \subseteq M$ not empty. For each $k \in \omega$ we define $T_{\Gamma,k}(M;X)$ as follows:*

$$\begin{aligned} T_{\Gamma,0}(M;X) \quad = \quad & \{c \in M : \text{ there exist } \varphi(x,y,\vec{z}) \in \Gamma, a, \vec{b} \in X \text{ such that} \\ & M \models \forall x \leq a\, \exists! y\, \varphi(x,y,\vec{b}) \text{ and there exists} \\ & d = \max\{e \in M : \ M \models \exists x \leq a\, \varphi(x,e,\vec{b})\} \text{ and } c \leq d\} \end{aligned}$$

$$\begin{aligned} T_{\Gamma,k+1}(M;X) \quad = \quad & \{c \in M : \text{ there exist } \varphi(x,y,\vec{z}) \in \Gamma, a, \vec{b} \in T_{\Gamma,k}(M;X) \text{ such that} \\ & M \models \forall x \leq a\, \exists! y\, \varphi(x,y,\vec{b}) \text{ and there exists} \\ & d = \max\{e \in M : \ M \models \exists x \leq a\, \varphi(x,e,\vec{b})\} \text{ and } c \leq d\} \end{aligned}$$

Then we define

$$T_\Gamma(M;X) = \bigcup_{k \in \omega} T_{\Gamma,k}(M;X)$$

If $\Delta_0 \subseteq \Gamma$, it is obvious that $X \subseteq T_{\Gamma,k}(M;X) \subseteq T_{\Gamma,k+1}(M;X)$.

Let $M \models d = (\max y)_{x \leq a}(\varphi(x,y,\vec{b}))$ denote $d = \max\{e \in M : M \models \exists x \leq a\, \varphi(x,e,\vec{b})\}$.

Note: If M is a model of certain Fragment of Arithmetic, then M verifies some maximum schemes that guarantee that all Γ-definable function with a nonempty and upper bounded domain have a maximum element ([4] and [5]).

Now we give some properties of this structures.

Proposition 3.2. *Let $M \models \mathbf{P}^-$, $\emptyset \neq X \subseteq M$ and $\Delta_0 \subseteq \Gamma$. Then $T_\Gamma(M;X) \subset M$.*

Proof.

It is sufficient to prove that given any two elements $c_1, c_2 \in T_\Gamma(M;X)$ we have that $c_1 + 1$, $c_1 + c_2$, $c_1 \cdot c_2 \in T_\Gamma(M;X)$.

Let $k \in \omega$ such that c_1, $c_2 \in T_{\Gamma,k+1}(M;X)$, then there exist $\varphi_1(x,y,\vec{z})$, $\varphi_2(x,y,\vec{z}) \in \Gamma$ and a, $\vec{b} \in T_{\Gamma,k}(M;X)$ such that

- $M \models \forall x \leq a\, \exists! y\, \varphi_1(x,y,\vec{b}) \wedge \exists d_1 = (\max y)_{x \leq a}(\varphi_1(x,y,\vec{b})) \wedge c_1 \leq d_1$.

- $M \models \forall x \leq a\, \exists! y\, \varphi_2(x,y,\vec{b}) \wedge \exists d_2 = (\max y)_{x \leq a}(\varphi_2(x,y,\vec{b})) \wedge c_2 \leq d_2$.

Suppose that $d_1 \geq d_2$ and consider $\theta(x,y,d_1) \equiv y = 2 \cdot d_1 \in \Delta_0$.

Let $e \in T_{\Gamma,k+1}(M;X)$. Then

- $M \models \forall x \leq e\, \exists! y\, \theta(x, y, d_1)$.

- $M \models 2 \cdot d_1 = (\max y)_{x \leq e}(\theta(x, y, d_1)) \wedge c_1 + c_2 \leq 2 \cdot d_1$.

So $c_1 + c_2 \in T_{\Gamma,k+2}(M; X) \subseteq T_\Gamma(M; X)$.
For $c_1 + 1$, $c_1 \cdot c_2 \in T_\Gamma(M; X)$ the proof is similar. □

Proposition 3.3. *Let $M \models \mathbf{P}^-$, $\emptyset \neq X \subseteq M$ and $\Gamma = \Sigma_n$ or Π_n. Then*

1. *For each $k \in \omega$ and for every $Y \subseteq T_{\Gamma,k}(M; X)$, $\mathcal{K}_n(M; Y) \subseteq T_{\Gamma,k+1}(M; X)$.*

2. *$\mathcal{K}_n(M; X) \subseteq T_\Gamma(M; X) = \mathcal{K}_n(M; T_\Gamma(M; X))$.*

Proof.

1. Let $a \in \mathcal{K}_n(M; Y)$.
   Then there exist $\varphi(x, \vec{y}) \in \Sigma_n$ and $\vec{b} \in Y$ such that $M \models \varphi(x, \vec{b}) \rightsquigarrow a$.
   We consider

   $\theta(x, y, \vec{z}) \equiv \varphi(y, \vec{z}) \in \Sigma_n$ if $\Gamma = \Sigma_n$ or
   $\theta(x, y, \vec{z}) \equiv \forall u\, (\varphi(u, \vec{z}) \rightarrow u = y) \in \Pi_n$ if $\Gamma = \Pi_n$ $(n > 0)$.

   Let $c \in X$. Then

   - $M \models \forall x \leq c\, \exists! y\, \theta(x, y, \vec{b})$.
   - $M \models a = (\max y)_{x \leq c}(\theta(x, y, \vec{b}))$.

   Hence $a \in T_{\Gamma,k+1}(M; X)$.

2. Taking into account that $X \subseteq T_{\Gamma,0}(M; X)$ we have by (1) that

   $$\mathcal{K}_n(M; X) \subseteq T_{\Gamma,1}(M; X) \subseteq T_\Gamma(M; X)$$

   Let $a \in \mathcal{K}_n(M; T_\Gamma(M; X))$.
   Then there exists $k \in \omega$ such that $a \in \mathcal{K}_n(M; T_{\Gamma,k}(M; X))$.
   Therefore, by (1), $a \in T_{\Gamma,k+1}(M; X) \subseteq T_\Gamma(M; X)$.

□

Proposition 3.4. *Let $M \models \mathbf{I}\Sigma_n$, $\emptyset \neq X \subseteq M$ and $\Gamma = \Sigma_{n+1}$ or Π_{n+1}. Then*

1. *$T_\Gamma(M; X) \prec^e_{n+1} M$.*

2. *If $T_\Gamma(M; X) \neq M$, then $T_\Gamma(M; X) \models \mathbf{B}\Sigma_{n+2}$.*

Proof.

1. By construction and (3.2) $T_\Gamma(M;X)$ is an initial substructure of M. Then we must see that $T_\Gamma(M;X) \prec_{n+1} M$.

   Let $\varphi(x,y) \in \Pi_n$ and $b \in T_\Gamma(M;X)$ such that $M \models \exists x\, \varphi(x,b)$. Since $\mathbf{I}\Sigma_n \iff \mathbf{L}\Pi_n$, there exists $c \in M$ such that $c = \min\{x \in M : M \models \varphi(x,b)\}$.

   We consider $\theta(x,y,z) \equiv \varphi(y,z) \wedge \forall u < y\, \neg\varphi(u,z) \in \Gamma(M)$.

   Since $b \in T_\Gamma(M;X)$, there exists $k \in \omega$ such that $b \in T_{\Gamma,k}(M;X)$.

   Let $a \in T_{\Gamma,k}(M;X)$. Then

   - $M \models \forall x \leq a\, \exists! y\, \theta(x,y,b)$.
   - $M \models c = (\max\, y)_{x \leq a}(\theta(x,y,b))$.

   Hence, $c \in T_{\Gamma,k+1}(M;X) \subseteq T_\Gamma(M;X)$ and $M \models \varphi(c,b)$.

   By Tarski-Vaught Test we obtain $T_\Gamma(M;X) \prec_{n+1} M$.

2. It follows from (1) and (1.5).

□

Proposition 3.5. *Let $M_1 \prec^e_{n+1} M_2 \models \mathbf{B}\Pi_n$, and $\emptyset \neq X \subseteq M_1$. Then*

1. $T_{\Sigma_{n+1}}(M_1;X) = T_{\Sigma_{n+1}}(M_2;X)$.

2. $T_{\Pi_n}(M_1;X) = T_{\Pi_n}(M_2;X)$.

Proof.

1. Let see by induction that for each $k \in \omega$

$$T_{\Sigma_{n+1},k}(M_1;X) = T_{\Sigma_{n+1},k}(M_2;X)$$

   $\boxed{k=0}$

   $\boxed{\subseteq}$ Let $c \in T_{\Sigma_{n+1},0}(M_1;X)$. Then there exist $\varphi(x,y,\vec{z}) \in \Sigma_{n+1}$, a, $\vec{b} \in X$ and $d \in M_1$ such that

   (i.) $M_1 \models \forall x \leq a\, \exists! y\, \varphi(x,y,\vec{b})$.
   (ii.) $M_1 \models d = (\max\, y)_{x \leq a}(\varphi(x,y,\vec{b}))$.
   (iii.) $M_1 \models c \leq d$.

   Since $M_1 \prec_{n+1} M_2$ and M_1, $M_2 \models \mathbf{B}\Pi_n$, those formulas are true in M_2. Hence, $c \in T_{\Sigma_{n+1},0}(M_2;X)$.

   $\boxed{\supseteq}$ Let $c \in T_{\Sigma_{n+1},0}(M_2;X)$. Then there exist $\varphi(x,y,\vec{z}) \in \Sigma_{n+1}$, a, $\vec{b} \in X$ and $d \in M_2$ such that

   (i.) $M_2 \models \forall x \leq a\, \exists! y\, \varphi(x,y,\vec{b})$.
   (ii.) $M_2 \models d = (\max\, y)_{x \leq a}(\varphi(x,y,\vec{b}))$.
   (iii.) $M_2 \models c \leq d$.

Let $e \in M_2$ such that $M_2 \models e \leq a \wedge \varphi(e, d, \vec{b})$. Since $M_1 \prec^e_{n+1} M_2$, we have that $e \in M_1$ and $M_1 \models \exists y\, \varphi(e, y, \vec{b})$.

Let $d' \in M_1$ such that $M_1 \models \varphi(e, d', \vec{b})$; then $M_2 \models \varphi(e, d', \vec{b})$, so $d = d' \in M_1$.

Since $M_1 \prec_{n+1} M_2$, formulas from (i) to (iii) are true in M_1.

Hence, $c \in T_{\Sigma_{n+1},0}(M_1; X)$.

$\boxed{k \to k+1}$

The proof is similar taking into account that by induction hypothesis

$$T_{\Sigma_{n+1},k}(M_2; X) = T_{\Sigma_{n+1},k}(M_1; X) \subseteq M_1$$

2. As in (1).

□

And then we get the main result of this paper, which gives us a satisfactorial answer to our questions.

4 Main Results

Theorem 4.1. *Let $M \models \mathbf{I}\Sigma_{n+2}$ and $X \subseteq M$ nonstandard and not cofinal in M. Then*

1. *$T_{\Sigma_{n+1}}(M; X)$ is an initial substructure, nonstandard and maximal Σ_{n+1}-definable in M that contains $\mathcal{K}_{n+1}(M; X)$.*

   Furthermore, this structure is the least verifying the properties above and

$$T_{\Sigma_{n+1}}(M; X) \models \mathbf{B}\Sigma_{n+2} \ + \ \neg\mathbf{I}\Sigma_{n+2}$$

2. *$T_{\Pi_n}(M; X)$ is an initial substructure, nonstandard and maximal Σ_n-definable in M containning $\mathcal{K}_n(M; X)$.*

Proof.

1. We have seen that

   (a) $T_{\Sigma_{n+1}}(M; X) \subset^e M$ (3.2).

   (b) $\mathcal{K}_{n+1}(M; X) \subseteq T_{\Sigma_{n+1}}(M; X) = \mathcal{K}_{n+1}(M; T_{\Sigma_{n+1}}(M; X))$ (3.3).

   Since $X \subseteq T_{\Sigma_{n+1}}(M; X)$, we have that $T_{\Sigma_{n+1}}(M; X)$ is nonstandard.

   So it remains to prove that $T_{\Sigma_{n+1}}(M; X) \neq M$.

   By (2.2) we have that $\mathcal{K}_{n+2}(M; X)$ is not cofinal in M, so there exists $a \in M$ such that $a > \mathcal{I}_{n+2}(M; X)$; thus, $\mathcal{I}_{n+2}(M; X) \prec^e_{n+1} M$ (2.3) and $\mathcal{I}_{n+2}(M; X) \neq M$. Then

$$T_{\Sigma_{n+1}}(M; X) \overset{(3.5)}{=} T_{\Sigma_{n+1}}(\mathcal{I}_{n+2}(M; X); X) \subseteq \mathcal{I}_{n+2}(M; X) \subsetneq M$$

- Let us see that $T_{\Sigma_{n+1}}(M;X)$ is the least structure verifying those properties.
  Consider $M' \subset^e M$ maximal Σ_{n+1}-definable in M such that $\mathcal{K}_{n+1}(M;X) \subseteq M'$.
  Let us see by induction that for each $k \in \omega$, $T_{\Sigma_{n+1},k}(M;X) \subseteq M'$.
  $\boxed{k=0}$
  Let $a \in T_{\Sigma_{n+1},0}(M;X)$. Then there exist $\varphi(x,y,\vec{z}) \in \Sigma_{n+1}$ and b, $\vec{c} \in X \subseteq M'$ such that

  - $M \models \forall x \leq b\, \exists! y\, \varphi(x,y,\vec{c})$.
  - $M \models a \leq m = (\max y)_{x \leq b}(\varphi(x,y,\vec{c}))$.

  Let $\theta(z,w,\vec{v}) \equiv \exists x \leq w\, \varphi(x,z,\vec{v}) \wedge \forall x \leq w\, \exists y\, (\varphi(x,y,\vec{v}) \wedge y \leq z) \in \Sigma_{n+1}(M)$.
  We have that $M \models \theta(z,b,\vec{c}) \rightsquigarrow m$ what implies that $m \in M'$, so $a \in M'$.
  $\boxed{k \to k+1}$
  The proof is similar taking into account that by induction hypothesis

$$T_{\Sigma_{n+1},k}(M;X) \subseteq M'$$

- $T_{\Sigma_{n+1}}(M;X) \models \mathbf{B}\Sigma_{n+2} + \neg\mathbf{I}\Sigma_{n+2}$.
  By (3.4), as $T_{\Sigma_{n+1}}(M;X) \neq M$, $T_{\Sigma_{n+1}}(M;X) \models \mathbf{B}\Sigma_{n+2}$
  Suppose that $T_{\Sigma_{n+1}}(M;X) \models \mathbf{I}\Sigma_{n+2}$.

  - By (3.4) $T_{\Sigma_{n+1}}(M;X) \prec^e_{n+1} M$.
  - By (3.5) $T_{\Sigma_{n+1}}(T_{\Sigma_{n+1}}(M;X);X) = T_{\Sigma_{n+1}}(M;X)$.

  But, from (1), it follows that $T_{\Sigma_{n+1}}(T_{\Sigma_{n+1}}(M;X);X)$ is a maximal Σ_{n+1}-definable set in $T_{\Sigma_{n+1}}(M;X)$. So $T_{\Sigma_{n+1}}(T_{\Sigma_{n+1}}(M;X);X) \neq T_{\Sigma_{n+1}}(M;X)$ what is a contradiction.

2. As in (1).

□

Note: This theorem cannot be improved because in the proof we have built a model M' ($M' = T_{\Sigma_{n+1}}(M;X)$), of $\mathbf{B}\Sigma_{n+2} + \neg\mathbf{I}\Sigma_{n+2}$ for which $T_{\Sigma_{n+1}}(M';X) = M'$.

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